GROUP PROPERTIES OF THE EQUATIONS OF ADIABATIC MOTION OF A MEDIUM IN RELATIVISTIC HYDRODYNAMICS

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A group classification is presented and the complete set of invariant solutions is found for the equations of adiabatic motion of a medium in relativistic hydrodynamics.

1. The group properties of the differential equations of hydrodynamics were studied in [1–3]; specifically, a group classification was made of the equations of adiabatic flow of a medium in the nonrelativistic case. It is of interest to apply the technique developed to the equations of relativistic hydrodynamics, since in view of their covariance the group properties of the latter must be different than in the nonrelativistic case. In the present paper we examine the group properties of the equations of adiabatic motion of a medium, since to the best of our knowledge their solutions have not yet been found.

In the general case the equations of adiabatic motion of a medium in relativistic hydrodynamics have the form

$$(S)\qquad \begin{gathered} u_i u_k \frac{\partial p}{\partial x_k} + \frac{\partial p}{\partial x_i} + W u_k \frac{\partial u_i}{\partial x_k} = 0 \qquad (W = p + \varepsilon) \\ \frac{\partial (n u_k)}{\partial x_k} = 0 \\ n u_k \frac{\partial p}{\partial x_k} - A u_k \frac{\partial n}{\partial x_k} = 0 \qquad \left(A(p, n) = -n \frac{\partial \sigma / \partial n}{\partial \sigma / \partial p}\right). \end{gathered} \qquad (1.1)$$

Here W is the enthalpy per unit volume of the medium, p is pressure, $\varepsilon = \varepsilon(p, n)$ is the internal energy density, n is the particle number density per unit volume, σ is the entropy per particle. The basic system (S) consists, respectively, of the equations of motion, particle number conservation, and entropy.

2. Let us study the group properties of (1.1). In accordance with the general rules [1–3] we must find the system of defining equations of the Lie algebra of the basic group G of the system given by (S).

The system of defining equations is found from the condition of invariance of the system (S) relative to the operator

$$X = \xi_x^1 \frac{\partial}{\partial x_1} + \xi_x^2 \frac{\partial}{\partial x_2} + \xi_u^1 \frac{\partial}{\partial u_1} + \xi_u^2 \frac{\partial}{\partial p} + \xi_u^3 \frac{\partial}{\partial n}.$$

Here we examine the case of one-dimensional motion and for symmetry of the form of the equations we set $t = x_2 (c = 1)$.

In simplified form the system of defining equations has the form

$$\begin{gathered} \frac{A}{n} \frac{\partial \xi_u^3}{\partial p} + \frac{\partial \xi_u^3}{\partial n} - \frac{1}{n} \xi_u^3 = 0 \\ \left(1 + \frac{\partial \varepsilon}{\partial p}\right) \xi_u^2 + \frac{\partial \varepsilon}{\partial n} \xi_u^3 - W \frac{\partial \xi_u^2}{\partial p} = 0 \\ \frac{\partial \ln A}{\partial p} \xi_u^2 + \frac{\partial \ln A}{\partial n} \xi_u^3 - \frac{\partial \xi_u^2}{\partial p} = 0, \qquad \xi_u^1 = \sqrt{u_1^2 + 1} \frac{\partial \xi_x^1}{\partial x_2} \\ \frac{\partial \xi_x^{(1,2)}}{\partial u_1} = \frac{\partial \xi_x^{(1,2)}}{\partial p} = \frac{\partial \xi_x^{(1,2)}}{\partial n} = 0, \qquad \frac{\partial \xi_u^{(2,3)}}{\partial u_1} = \frac{\partial \xi_u^1}{\partial p} = \frac{\partial \xi_u^1}{\partial n} = \frac{\partial \xi_u^2}{\partial n} = 0 \\ \frac{\partial \xi_x^1}{\partial x_{(1,2)}} = \frac{\partial \xi_x^2}{\partial x_{(2,1)}}, \quad \frac{\partial^2 \xi_x^2}{\partial x_1 \partial x_2} = \frac{\partial^2 \xi_x^1}{\partial x_1^2} = \frac{\partial^2 \xi_x^1}{\partial x_2^2}, \quad \frac{\partial^2 \xi_x^1}{\partial x_1 \partial x_2} = \frac{\partial^2 \xi_x^2}{\partial x_1^2} = \frac{\partial^2 \xi_x^2}{\partial x_2^2} \\ n \frac{\partial \xi_u^2}{\partial x_{(1,2)}} - A \frac{\partial \xi_u^3}{\partial x_{(1,2)}} = 0, \quad \frac{\partial^2 \xi_x^1}{\partial x_2^2} + \frac{1}{W} \frac{\partial \xi_u^2}{\partial x_1} = 0, \quad \frac{\partial^2 \xi_x^1}{\partial x_1 \partial x_2} + \frac{1}{W} \frac{\partial \xi_u^2}{\partial x_2} = 0 \\ \frac{\partial^2 \xi_x^1}{\partial x_2^2} + \frac{1}{n} \frac{\partial \xi_u^3}{\partial x_1} = 0, \qquad \frac{\partial^2 \xi_x^1}{\partial x_1 \partial x_2} + \frac{1}{n} \frac{\partial \xi_u^3}{\partial x_2} = 0. \end{gathered} \qquad (2.1)$$

The following equation can be derived as a corollary from the system of defining equations (2.1):

$$\left(1-\frac{A}{W}\right)\frac{\partial \xi_u^{(2,3)}}{\partial x_{(1,2)}}=0$$

from which it follows that two cases are possible, either A = W or

$$\frac{\partial \xi_u^{(2,3)}}{\partial x_{(1,2)}}=0 .$$

In the first case the basic system ac 'ts an infinite group, since $\zeta_x^{(1,2)}$ are expressed through arbitrary wave functions. In the second case $\zeta_x^{(2,3)}$ are independent of x and, consequently, $\zeta_x^{(1,2)}$ depends on x to the first power. In the following we assume that $A \neq W$, i.e., we examine the second case.

For the group classification of the system (S) we must find the specialization of the system as a consequence of A and ε. The functions A and ε obey the system of equations

$$\frac{\partial F}{\partial n}=0, \qquad -\frac{\partial}{\partial n}\left(\frac{1}{Q}\right)+\frac{\partial}{\partial n}\left(\frac{1}{Q}\frac{\partial Q}{\partial p}\right)+\frac{\partial}{\partial n}\left(\frac{n}{AQ}\frac{\partial Q}{\partial n}\right)=0 \tag{2.2}$$

where

$$W\frac{\partial \ln A}{\partial n}-\frac{\partial \varepsilon}{\partial n}=F(p)\left[\frac{\partial \ln A}{\partial n}\left(1+\frac{\partial \varepsilon}{\partial p}\right)-\frac{\partial \ln A}{\partial p}\frac{\partial \varepsilon}{\partial n}\right]$$
$$Q\left(W\frac{\partial \ln A}{\partial n}-\frac{\partial \varepsilon}{\partial n}\right)=\left(1+\frac{\partial \varepsilon}{\partial p}\right)-W\frac{\partial \ln A}{\partial p} .$$

In particular, it can be shown that (2.2) admits a solution of the form

$$\varepsilon = a_1 p^{a_2} n^{a_3} + a_1^1 p^{a_2^1} n^{a_3^1}, \quad A = b_1 p$$
$$(a_1, a_2, a_3, a_1^1, a_2^1, a_3^1, b_1 = \text{const})$$

For $b_1=\gamma$, $a_1=1/(\gamma-1)$, $a_2=a_3^1=1$, $a_3=0$, $a_1^1=m_0$, where m_0 is the rest mass of the particles and γ is the adiabatic exponent, we have the equation of state of the relativistic ideal gas

$$\varepsilon=\frac{p}{\gamma-1}+nm_0, \qquad A=\gamma p . \tag{2.3}$$

For the ultrarelativistic case

$$\varepsilon = p/(\gamma-1) \qquad (p, \varepsilon \gg nm_0) .$$

This corresponds to high densities of the pressure and internal energy.

The general solution of (2.1) for the equation of state (2.3) has the form

$$\xi_x^1 = l_1x_1 + l_2x_2 + l_3, \qquad \xi_x^2 = l_2x_1 + l_1x_2 + l_4 ,$$
$$\xi_u^1 = l_2\sqrt{u_1^2+1}, \quad \xi_u^2 = l_5 p, \quad \xi_u^3 = l_5 n \quad (l_1, \ldots, l_5 = \text{const}). \tag{2.4}$$

Hence we find the basis operators of the Lie algebra for the basic group G

$$X_1 = x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}+\sqrt{u_1^2+1}\,\frac{\partial}{\partial u_1}, \quad X_2 = p\frac{\partial}{\partial p}+n\frac{\partial}{\partial n}$$
$$X_3 = x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}, \quad X_4=\frac{\partial}{\partial x_1}, \quad X_5=\frac{\partial}{\partial x_2} . \tag{2.5}$$

As is known, to find essentially different particular solutions it is necessary to construct an optimum system of single parameter subgroups of the group G of the basic system (S).

The optimum system is constructed by using the internal automorphisms of the group G and has the following form:

$$H_1 = X_4 - X_5, \quad H_2 = X_3, \quad H_3 = X_2 - X_4 + X_5,$$
$$H_4 = X_1, \quad H_5 = X_2 + X_3, \quad H_6 = X_1 + \alpha X_3,$$
$$H_7 = X_1 + X_2, \quad H_8 = X_1 + X_2 + \alpha X_3, \qquad (\alpha \neq 1). \tag{2.6}$$

The eight operators (2.6) lead to invariant solutions of rank one.

Before turning to the study of these invariant solutions, we must emphasize that an exact analytic expression for the solutions is possible only in the ultrarelativistic case. In the relativistic case the solution is found by successive approximations, by expanding in powers of the constant m_0. We shall make the analysis of the solutions only for the ultrarelativistic case.

Subgroup H_1. The operator invariants are

$$I_1 = \lambda = x_1 + x_2, \qquad I_2 = V(\lambda) = u_1, \; I_3 = P(\lambda) = p, \; I_4 = R(\lambda) = n.$$

The system (S/H) of ordinary differential equations has a solution only with constant values of the velocity, pressure, and density, i.e., the solution of this subgroup describes an infinite and homogeneous medium.

Subgroup H_2. The invariants of the operator X_3 are

$$I_1 = \lambda = \frac{x_1}{x_2}, \quad I_2 = V(\lambda) = u_1, \quad I_3 = P(\lambda) = p, \quad I_4 = R(\lambda) = n. \tag{2.7}$$

The system of equation (S/H) has the solution

$$V(\lambda) = \frac{\lambda - \sqrt{\gamma - 1}}{\sqrt{2-\gamma}\,\sqrt{1-\lambda^2}}, \quad \frac{P}{P_0} = \left(\frac{R}{R_0}\right)^{\gamma}, \quad R = R_0 \exp\left\{\frac{1}{\sqrt{\gamma-1}} \arcsin \lambda\right\} \quad (\lambda \leqslant 1).$$

For the velocity of the medium we have the usual formula for relativistic addition of the velocities

$$v = \frac{\lambda - \sqrt{\gamma-1}}{1 - \lambda\sqrt{\gamma-1}}$$

where $\sqrt{\gamma - 1}$ is the traveling wave propagation velocity.

Subgroup H_3. The operator invariants are

$$I_1 = \lambda = x_1 + x_2, \; I_2 = p^{-1} \exp\{-{}^1/_2\,(x_1 - x_2)\} = P, \; I_3 = p\,/\,n = R, \; I_4 = u_1 = V.$$

The corresponding equations (S/H) can be integrated and we have for the invariants

$$\frac{P}{P_0} = \lambda_0^{\theta_1}, \quad \frac{R}{R_0} = \lambda_0^{\theta_2}, \quad V = \frac{\lambda_0 - 1}{2\sqrt{\lambda_0}}$$
$$\left(\lambda_0 = -2\beta\lambda + c_1, \quad \beta = \frac{2(\gamma-1)}{\gamma(2-\gamma)}, \quad \theta_1 = \frac{\gamma(4-\gamma)}{4(\gamma-1)}, \quad \theta_2 = \frac{1-\gamma}{2}\right).$$

Here there is no polytropic dependence between the density and pressure and, consequently, the entropy σ is not constant.

Subgroup H_4. The invariants of the operator X_1 are

$$I_1=\lambda=\sqrt{x_2^2-x_1^2},\quad I_2=V(\lambda)=(x_2-x_1)(u_1+\sqrt{u_1^2+1}) \tag{2.8}$$
$$I_3=P(\lambda)=p,\ I_4=R(\lambda)=n.$$

The system of differential equations (S/H) has the form

$$\frac{\gamma}{(\gamma-1)}P\left[V'\frac{(V^2+\lambda^2)}{V\lambda}-2\right]+P'\frac{(V^2-\lambda^2)}{\lambda}=0$$
$$R\left[V'\frac{(V^2-\lambda^2)}{V\lambda}+2\right]+R'\frac{(V^2+\lambda^2)}{\lambda}=0\qquad\left(\frac{P'}{P}=\gamma\frac{R'}{R}\right). \tag{2.9}$$

The invariant V satisfies the equation

$$\lambda^2\left(\frac{V^2}{\lambda^2}+1\right)^{4(\gamma-1)/(\gamma-2)}\left(\frac{V^2}{\lambda^2}-1\right)^{2/(2-\gamma)}V^2=c_1\exp\left(-\frac{\lambda^4}{2}\right).$$

Specifically, for $\gamma=4/3$ the invariant V is defined by a cubic equation. The operator X_1 can be treated as a dilation operator.

Subgroup H_5. The invariants

$$I_1=\lambda=\frac{x_1}{x_2},\quad I_2=P(\lambda)=\frac{x_1-x_2}{p},\quad I_3=R(\lambda)=\frac{p}{n},\quad I_4=V(\lambda)=u_1$$

lead to the system (S/H), but their solutions cannot be found in analytic form. Here again there is no polytropic dependence between the density and pressure.

Subgroup H_6. In this case we have a combination of two operators examined above, namely, the dilation operator X_1 and the simple wave operator X_3. The operator invariants

$$I_1=\lambda=\frac{(x_1-x_2)^{k+1/2}}{(x_1+x_2)^{k-1/2}},\quad I_2=V(\lambda)=(u_1+\sqrt{u_1^2+1})(x_1-x_2)^{1/(1-2k)}$$
$$I_3=P(\lambda)=p,\ I_4=R(\lambda)=n,\ k=1/2\alpha \tag{2.10}$$

satisfy a system (S/H) of the form

$$\frac{\gamma}{(\gamma-1)}\frac{V'}{V}[(1-2k)V^2-(1+2k)\lambda^{\varkappa}]+$$
$$+\frac{P'}{P}[(1-2k)V^2+(1+2k)\lambda^{\varkappa}]+\frac{2\gamma}{(\gamma-1)(1-2k)}\lambda^{\varkappa-1}=0$$
$$\frac{V'}{V}[(1-2k)V^2+(1+2k)\lambda^{\varkappa}]+\frac{R'}{R}[(1-2k)V^2-(1+2k)\lambda^{\varkappa}]-\frac{2}{(1-2k)}\lambda^{\varkappa-1}=0 \tag{2.11}$$
$$\frac{P'}{P}=\gamma\frac{R'}{R},\quad \varkappa=\frac{2}{1-2k},\quad k\neq\frac{1}{2}.$$

Let us examine in more detail the solution of these equations, since here there is a polytropic dependence between the density and pressure, which is important for applications. In order to integrate (2.11) we set

$$V=\lambda^{(1-2k)^{-1}}\varphi(\lambda). \tag{2.12}$$

It is not possible to express all the invariants V, R, P directly through a known function of the variable λ. Therefore it is more convenient to represent the invariants in terms of the new variable $y=\varphi^2$. In these variables the integrals of (1.11) take the form

$$
\begin{gathered}
R / R_0 = (y - \alpha_1)^{D_1} (y + \alpha_2)^{D_2} [y - (b_4 + b_3)]^{D_3} [y - (b_4 - b_3)]^{D_4} \\
V = c^{1/(1-2k)} y^{1/2(1-A)} (y - \alpha_1)^{-1/2 B} (y + \alpha_2)^{-1/2 C} \\
\lambda / c_1 = y^{-1/2 A(1-2k)} (y - \alpha_1)^{-1/2 B(1-2k)} (y + \alpha_2)^{-1/2 C(1-2k)} \\
P / P_0 = (R / R_0)^{\gamma}
\end{gathered}
\tag{2.13}
$$

where

$$
\begin{gathered}
D_1 = -\frac{2A_1 B}{(2-\gamma)(1-2k)}, \quad D_2 = -\frac{2A_2 C}{(2-\gamma)(1-2k)} \\
D_3 = -\frac{2}{(2-\gamma)(1-2k)} \left[\frac{A}{2b_3} + BB_1 + CB_2 \right], \\
D_4 = -\frac{2}{(2-\gamma)(1-2k)} \left[-\frac{A}{2b_3} + BC_1 + CC_2 \right] \\
A = \frac{b_1^2 + 2a_1 b_1 + a_2}{b_1^2 - b_2^2}, \quad B = \frac{b_2^2 - 2a_1 b_2 + a_2}{2b_2(b_1 + b_2)}, \quad C = \frac{b_2^2 + 2a_1 b_2 + a_2}{2b_2(b_2 - b_1)} \\
A_1 = \frac{\alpha_1}{(b_4 - \alpha_1)^2 - b_3^2}, \quad B_1 = \frac{b_4 + b_3}{2b_3(b_3 + b_4 - \alpha_1)}, \quad C_1 = \frac{b_4 - b_3}{2b_3(b_3 - b_4 + \alpha_1)} \\
A_2 = -\frac{\alpha_2}{(b_4 + \alpha_2)^2 - b_3^2}, \quad B_2 = \frac{b_4 + b_3}{2b_3(b_3 + b_4 + \alpha_2)}, \quad C_2 = \frac{b_4 - b_3}{2b_3(b_3 - b_4 + \alpha_2)} \\
a_1 = \frac{\gamma}{(2-\gamma)(1-2k)}, \quad a_2 = \frac{4(1-\gamma)(1+2k)^2 + \gamma^2}{(2-\gamma)^2(1-2k)^2} \\
b_1 = \frac{2\gamma k}{(2-\gamma)(1-2k)}, \quad b_2 = \frac{2k}{(1-2k)} \left[\left(\frac{\gamma}{2-\gamma} \right)^2 + \frac{1-4k^2}{4k^2} \right]^{1/2} \\
b_3 = \left(\frac{1+2k}{1-2k} \right) \left[\left(\frac{\gamma}{2-\gamma} \right)^2 - 1 \right]^{1/2}, \quad b_4 = \frac{\gamma(1+2k)}{(2-\gamma)(1-2k)} \\
\alpha_1 = b_2 + b_1, \quad \alpha_2 = b_2 - b_1.
\end{gathered}
$$

Subgroup H_7. The invariants of this subgroup

$$
\begin{gathered}
I_1 = \lambda = \sqrt{x_2^2 - x_1^2}, \quad I_2 = V(\lambda) = (u_1 + \sqrt{u_1^2 + 1})(x_2 - x_1) \\
I_3 = P(\lambda) = (u_1 + \sqrt{u_1^2 + 1}) / p, \quad I_4 = R(\lambda) = p / n
\end{gathered}
$$

satisfy the system (S/H) of equations, which can be integrated by the same method. Setting $V = \lambda\varphi$, $y = \varphi^2$, we obtain the solution in the form

$$
\begin{gathered}
P / P_0 = (y - 1)^{A_1} (y - \alpha_1)^{A_2} (y - \alpha_2)^{A_3} (y - \beta_1)^{A_4} (y - \beta_2)^{A_5} \\
R / R_0 = y^{B_1} (y - 1)^{B_2} (y + 1)^{B_3} (y - \alpha_1)^{B_4} (y - \alpha_2)^{B_5} (y - \beta_1)^{B_6} (y - \beta_2)^{B_7} \\
\lambda / c_1 = y^{E_1} (y + \delta_1)^{E_2} (y + \delta_2)^{E_3}.
\end{gathered}
\tag{2.14}
$$

The constants, other than the constants of integration P_0, R_0, c_1, depend on the adiabatic exponent γ.

Subgroup H_8. This subgroup differs from subgroup H_6 in the addition of the operator X_2. The invariants

$$
\begin{gathered}
I_1 = \lambda = \frac{(x_1 - x_2)^{k+1/2}}{(x_1 + x_2)^{k-1/2}}, \quad I_2 = (u_1 + \sqrt{u_1^2 + 1})^{1-2k} (x_1 - x_2) \\
I_3 = P(\lambda) = (u_1 + \sqrt{u_1^2 + 1}) / p, \quad I_4 = R(\lambda) = p / n, \quad k = 1/2\,\alpha
\end{gathered}
$$

also lead to the system (S/H), which are integrated similarly to the case of subgroups H_6 and H_7. The solution in general form is

$$
\begin{gathered}
\frac{P}{P_0} = y^{A_1} \left[y + \frac{(1+2k)}{(1-2k)} \right]^{A_2} (y - \alpha_1)^{A_3} (y - \alpha_2)^{A_4} (y - \beta_1)^{A_5} (y - \beta_2)^{A_6} \\
\frac{R}{R_0} = y^{B_1} \left[y + \frac{(1+2k)}{(1-2k)} \right]^{B_2} \left[y - \frac{(1+2k)}{(1-2k)} \right]^{B_3} \left[y + \frac{(\gamma-1)(1+2k)}{(2\gamma+1)(1-2k)} \right]^{B^4} \times \\
\times (y - \alpha_1)^{B_5} (y - \alpha_2)^{B_6} (y - \beta_1)^{B_7} (y - \beta_2)^{B_8} \\
\lambda / c_1 = (y - a)^{E_1} (y - c)^{E_2} (y + c)^{E_3}.
\end{gathered}
\tag{2.15}
$$

Here all the constants depend on k and γ.

Thus, with the exception of all the subgroups in which the operator X_2 is missing, there is no polytropic dependence between the density and pressure. At the present time the subgroup H_6 is of the greatest interest, since here the entropy is conserved and therefore the resulting solutions can be applied, for example, to the hydrodynamic theory of multiple formation of particles [4–5].

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